Noise-enhanced phase locking in a stochastic bistable system driven by a chaotic signal

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We study the dynamics of an overdamped bistable system driven simultaneously by noise and a chaotic input signal. The effect of synchronization of switchings in a stochastic bistable system by a subthreshold chaotic signal is found and described in terms of the theory of phase synchronization. Using the two different definitions of the instantaneous phase of stochastic and chaotic oscillations, we show explicitly the effect of noise-enhanced phase locking both for coherent and for broadband chaotic input signals. The quantitative analysis of this effect has shown that the degree of phase coherence estimated by means of the effective diffusion constant is maximal in some range of noise intensities. Moreover, there is a synchronization region on the parameter plane ''amplitude of chaotic signal–noise intensity'' in which the phases and mean frequencies of an input signal and of response are locked. [S1063-651X(99)09301-0]

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I. INTRODUCTION

Synchronization is a basic phenomenon in physics and the classical example of self-organization in nonlinear oscillatory systems. To synchronize means to concur or agree in time, to proceed or to operate at exactly the same rate, to happen at the same time [1]. The phenomenon of synchronization occurs in nonlinear self-sustained oscillators driven by external periodic force or coupled with each other [2,3]. In general, synchronization can be treated as the appearance of some functionals characterizing the correlations in temporal behavior of two or more processes [4]. The instantaneous phase plays the role of such a functional in the classical theory of oscillations. Synchronization is defined in this case as the locking of phases of the self-sustained oscillator $\Phi(t)$ and of external periodic force $\Psi(t)=\Omega_0 t$: $|n\Phi(t)-m\Psi(t)|<\text{const}$, or by a weaker condition of frequency locking $\Omega=\dot{\Phi}=(m/n)\,\Omega_0$; here n,m are some integers. The above conditions are fulfilled in finite regions of the parameter space of the system which are called Arnold tongues. Recently, the classical approach to synchronization based on the notion of an instantaneous phase of oscillations was generalized in the cases of nonautonomous and interacting chaotic systems [5,6].

Synchronization of self-sustained oscillators in the presence of noise was considered for the first time in [7] and was then studied in detail by Stratonovich [8]. As was shown, Gaussian noise leads to fluctuations of the phase difference $\phi(t)=\Phi(t)-\Psi(t)$ and, in the assumption of constant amplitude, its slow dynamics can be described by the following stochastic differential equation (SDE):

$$\dot{\phi}=\Delta-\epsilon G(\phi)+\xi(t), \qquad (1)$$

where $\Delta=\Omega-\Omega_0$ and ϵ are the frequency mismatch and the parameter of nonlinearity, respectively, $G(\phi)$ is a 2π periodic function, and $\xi(t)$ is Gaussian noise [8]. In the case of a Van der Pol oscillator driven by an external periodic force, $G(\phi)\equiv\sin\phi$ and the phase difference performs overdamped Brownian motion in the tilted periodic potential $U(\phi)=-\Delta\phi-\epsilon\cos\phi$. In the case of $\Delta<\epsilon$ and of small noise intensity, this motion can be divided into two parts: the phase difference fluctuates for a long time inside a well of the potential $U(\phi)$ (the phase locking) and rarely makes jumps from one potential well to another (i.e., displays phase slips).

As is well known, the effect of noise on the synchronized self-sustained oscillator is negative: the increase of noise intensity leads to the loss of phase coherence (phase slips become more frequent) and shrinks Arnold tongues [9]. However, there are qualitatively different situations when the noise plays a constructive role [10]. One of the typical examples of the positive effect of noise is stochastic resonance (SR) [11–14], which occurs in a wide class of nonlinear systems driven simultaneously by noise and a signal. To demonstrate SR, a nonlinear system should possess a noise-controlled time scale. Traditionally, SR is described in terms of the spectral power amplification [15] and signal-to-noise ratio [16]. The dependencies of these characteristics versus noise intensity have a bell-shaped maximum that allows us to determine SR as amplification of a weak signal applied to the input of the system by tuning the noise level.

Another approach to the description of SR is based on the statistics of residence times [17,18]. The residence times distribution has an exponential shape in the absence of an input signal. In the presence of a weak signal this distribution is structurized and contains a series of peaks centered at odd multiples of the half-period of the signal [12]. As was shown in recent work [19], to get the correct results it is necessary to analyze the difference between the residence-time distribution in the presence of modulation and the residence-time distribution in the absence of modulation. The deviation from undriven residence-time distribution at the time of the half-period of the external signal can be used as the measure for SR.

It is important to underline that, usually, an amplitude of periodic force takes much less than a barrier separating the potential wells and SR can be correctly described by linear-response theory [20–22]. In this case the response of the stochastic resonator is fully determined by the linear susceptibility of the system and the structure of the input signal is immaterial: the signal can be harmonic, quasiperiodic [23], or even aperiodic broadband noisy [22,24]. As was shown in [25], noise can enhance the response of a nonlinear system to a weak input signal, regardless of whether the signal is periodic or aperiodic. The cross-correlation functions and transinformation that directly quantifies the rate of information transfer through a stochastic system were used to quantitatively characterize this phenomenon, which was called aperiodic stochastic resonance (ASR). ASR in excitable systems driven by a chaotic input signal was considered in [26]. It was shown that the information transfer between the two relaxation-type nonlinear oscillators is optimized by intermediate noise levels.

It is well known that in some cases the degree of coherence between switchings in a stochastic bistable system and an external signal may be sufficiently large [11]. It allows us to speak about synchronization features of SR, which became the subject of a lot of theoretical and experimental investigations [12,22]. These features manifest themselves more brightly in situations when the amplitude of the periodic force is large enough, although it is insufficient to cause the switchings in the absence of noise. As was shown in [27], the effect of mean switching frequency locking in a Schmitt trigger driven simultaneously by noise and periodic signal takes place. Moreover, as follows from the results of [28], synchronization of the switching processes can also be observed in the case of the absence of deterministic time scales when the interaction of statistical time scales of subsystems takes place. As follows from these results that SR systems can demonstrate synchronizationlike phenomena which are similar to the classical synchronization mentioned above.

The classical notion of synchronization and mentioned synchronizationlike effects occurring in periodically driven stochastic bistable systems [27] were discussed in [29]. It has been shown that for a sufficiently large amplitude of periodic force the noise-enhanced phase locking takes place and SR, in this case, can be correctly described in terms of the classical theory of oscillations. However, in [29] the case of a periodic input signal was considered only, whereas the real-world signals are often not periodic and contain both phase and amplitude fluctuations.

The main goal of the present study is to take the next step in the study of synchronization in stochastic bistable systems and to generalize the approach proposed in [29] to the more complex and interesting case of chaotic input signals. Such signals can be considered as a simple model of real signals, which in spite of their deterministic nature, contain both amplitude and phase fluctuations. The previous study of the passing of nonperiodic signals through the stochastic bistable system dealt with a consideration of the influence of the amplitude and phase fluctuations on the SR characteristics (amplification and signal-to-noise ratio) [30,31]. However, these results do not provide any information about instantaneous matching of input signals and of output switching events. In other words, we want to find the answers to the following questions: Is it possible to observe synchronization between a chaotic input signal and a response in the classical sense of the coincidence of their instantaneous phases? If yes, then how long will they remain synchronized? To answer these questions, we introduce in Sec. II the instantaneous phase of stochastic and chaotic oscillations based on the analytic signal concept and demonstrate the noise-enhanced phase locking for the input signal from the Rössler system. Section III is devoted to the study of synchronization of a switching process in a stochastic bistable system by the broadband chaotic signal from the Lorenz system. Our conclusions are given in Sec. IV.

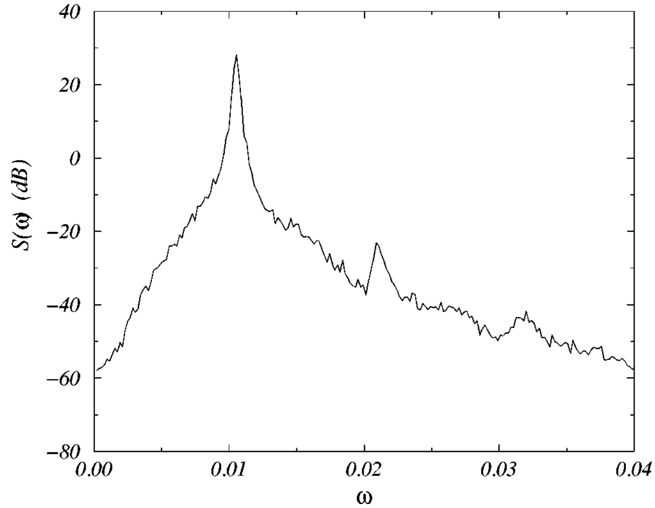

FIG. 1. The power spectrum of oscillations in the Rössler system for $c=7.1$. The largest Lyapunov exponent is $\lambda=0.066\,58$.

II. NOISE-ENHANCED INSTANTANEOUS PHASE LOCKING

We treat as a model an overdamped stochastic bistable system driven by the slowly varying chaotic signal, which is described by the following SDE:

$$\dot{x}=\alpha x-\beta x^3+\sqrt{2D}\xi(t)+k\mu(t), \tag{2}$$

where $\xi(t)$ is white Gaussian noise, $\alpha,\beta>0$, k is a some constant, and $\mu(t)$ is the signal from a chaotic system.

First, let us consider the case of the chaotic input signal from the Rössler system, which is described by the following ordinary differential equations: $\dot{x}_1=-\tau(x_2+x_3)$, $\dot{x}_2=\tau(x_1+0.15x_2)$, $\dot{x}_3=\tau[0.2+x_3(x_1-c)]$, where $\tau=0.01$ is the small parameter which was introduced in the model to reduce the characteristic time scale and $c>6.1$. It is known that the Rössler system for chosen values of the parameters demonstrates the regime of so-called weak chaos which is characterized by the presence of a sharp single peak in the power spectrum of oscillations (see Fig. 1). That gives the possibility to consider the oscillations in the Rössler system as close to periodic with slowly varying amplitude and phase (we can introduce into consideration the quasiperiod of chaotic oscillations T which equals to the mean return time in the secant plane $x_1=0$). This fact and also the presence of the direction in the phase space of a chaotic system which corresponds to the zero Lyapunov exponent give the possibility to introduce the phase of chaotic oscillations and to describe the external and mutual chaotic synchronization in terms of the classical theory of phase synchronization [6]. According to [5,6], the instantaneous phase of chaotic oscillations can be introduced in three ways: through the return times in some secant plane, through a phase space projection, and through the analytic signal concept. According to the approach proposed in [29], we will define the instantaneous phases of both the input signal from the Rössler system and the response of the stochastic bistable system by means of the analytic signal concept, based on the Hilbert transformation (HT). The analytic signal $w(t)$ is a complex function of time defined as

$$w(t)=x(t)+iy(t)=A(t)e^{i\Phi(t)}, \quad (3)$$

where $y(t)$ is the HT of original process $x(t)$:

$$y(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau. \quad (4)$$

In the last expression the integral is taken in the sense of the Cauchy principal value. Instantaneous amplitude $a(t)$ and phase $\Phi(t)$ of $x(t)$ are unambiguously defined through this concept as

$$\Phi(t)=\arctan\left[\frac{y(t)}{x(t)}\right], \quad a^2(t)=x^2(t)+y^2(t). \quad (5)$$

The mean frequency $\langle\omega\rangle$ is then defined as

$$\langle\omega\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T\omega(t)dt. \quad (6)$$

As was mentioned above, noise shrinks Arnold tongues of the self-sustained oscillator and makes obscure the classical definition of synchronization. Evidently, in this case we can speak about *effective synchronization* only [32]. To define the conditions of effective synchronization of a noisy dynamical system we must impose some restrictions either on phase (frequency) fluctuations or on the output signal-to-noise ratio. We will consider the strongest definition based on statistics of phase fluctuations.

The instantaneous phase difference between the output and input signals is

$$\phi(t)=\Phi(t)-\Psi(t), \quad (7)$$

where $\Phi(t)$ is the phase of the stochastic bistable system and $\Psi(t)$ is the phase of the chaotic signal $x_1(t)$ from the Rössler system. To determine $\phi(t)$, we have integrated numerically the original SDE (2) and then we perform HT using a well-established technique (see, for example, [33]). The results of our calculations are reported in Fig. 2. The effect of *noise-enhanced phase locking* is clearly seen. The instantaneous phases of the output and input signals are locked at an optimal noise level in the stochastic resonator and their difference fluctuates about some constant value. As the noise intensity deviates from an optimal value, the duration of the time intervals in the course of which the phases are locked decreases and phase slips appear, so that we can speak about partially synchronized phase dynamics. It is remarkable that the dynamics of the phase difference $\phi(t)$ as well as in the case of periodic driving [29] is very similar to that of a synchronized self-sustained oscillator and can be qualitatively described by SDE (1). The dependence of mean frequency (6) and mean switching frequency [27,29] versus noise intensity is presented in Fig. 3. As clearly seen, there is

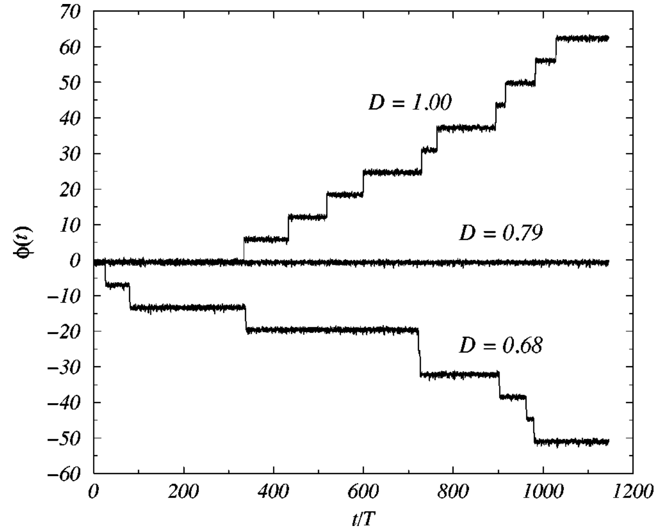

FIG. 2. The instantaneous phase difference versus time (in units of quasiperiod of the chaotic signal) for indicated values of noise intensity D. Other parameters are $\alpha=5$, $\beta=1$, $c=7.1$, $k=0.35$, $\mu(t)=x_1(t)-1.025\,18$.

the range of noise intensities in which the mean frequency of stochastic oscillations coincides with the mean frequency of chaotic oscillations, e.g., the *locking of mean frequency of chaotic oscillations* takes place. The results reported in Figs. 2 and 3 are very similar to those obtained in [29], where the case of periodic forcing was considered. This fact means that the phase fluctuations contained in the input signal are passed through the stochastic resonator without any distortions at an optimal noise level. Although the above results clearly display the effect of phase synchronization, we need to estimate the duration of phase locking times to determine the effective synchronization of a stochastic system.

According to the definition of effective synchronization, a stochastic bistable system driven by the chaotic signal can be

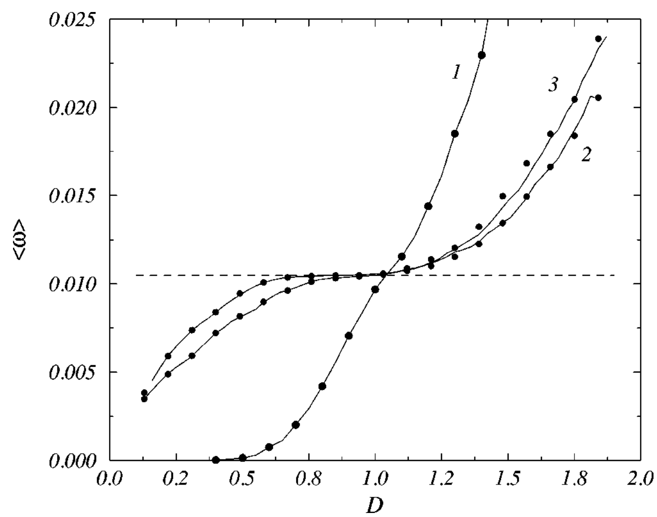

FIG. 3. The dependence of mean frequency (6) (solid line) and of mean switching frequency (symbols) versus noise intensity for different values of parameters c and k: (1) $k=0$; (2) $c=7.1$, $k=0.35$; (3) $c=8.5$, $k=0.30$. Dashed line is the value of the mean frequency of chaotic oscillations in the Rössler system. Other parameters are the same as in the previous figure.

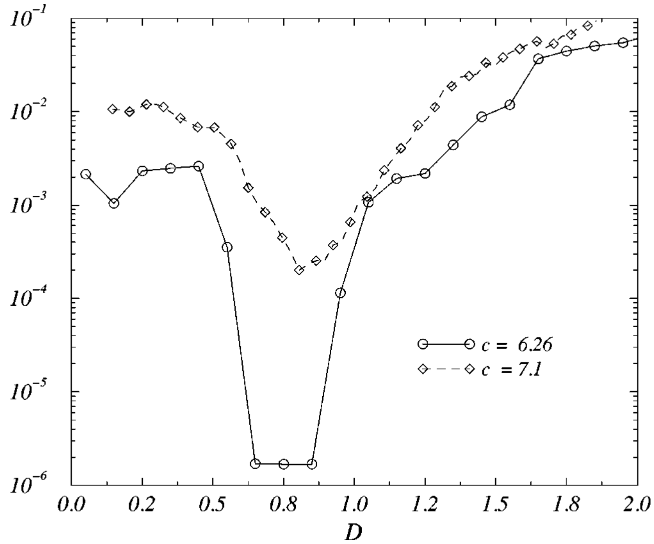

FIG. 4. The effective diffusion constant D_{eff} vs noise intensity for indicated values of parameter c in the Rössler system. Other parameters are the same as in Fig. 2.

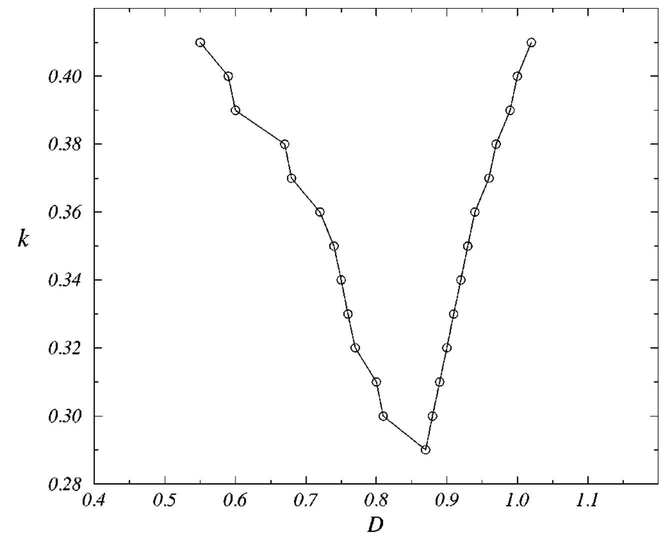

FIG. 5. Synchronization region on the parameter plane ''noise intensity-amplitude of chaotic signal.'' Other parameters are $\alpha=5$, $\beta=1$, $c=6.26$, $\mu(t)=x_1(t)$.

considered as effectively synchronized if the mean time in the course of which the instantaneous phase of the system is locked is much larger than the characteristic time scale of a chaotic system. There is no universal and strict definition of a time scale of a chaotic dynamical system at present. Its definition is a separate task in each concrete case. For the input signal from the Rössler system demonstrating the regime of weak chaos, it is reasonable to consider the quasiperiod of chaotic oscillations as such a time scale. In our study we will consider the chaotically driven overdamped Kramers oscillator (2) to be effectively synchronized if its phase appears to be locked in the course of a time that is much larger than the quasiperiod of oscillations in the Rössler system. As seen from Fig. 2, the phase of stochastic oscillations is locked in the course of the hundreds of quasiperiods of chaotic signal, which means synchronization in the above defined sense. To estimate a duration of phase locking time intervals we will use the *effective diffusion constant*, which is defined by the following relation:

$$D_{\mathrm{eff}}=\frac{1}{2}\,\frac{d}{dt}[\langle \phi^2(t)\rangle-\langle \phi(t)\rangle^2]. \tag{8}$$

This quantity was originally introduced in the classical works [8,32] and characterizes the spreading of an initial distribution of the phase difference along the potential profile $U(\phi)$. It can be shown that the effective diffusion constant is proportional to the mean escape rate r from a well of the potential $U(\phi)$: $D_{\mathrm{eff}}=4\pi^2 r$ [8], i.e., D_{eff} is inversely proportional to the mean time interval of phase locking. The dependence of the effective diffusion constant (8) versus noise intensity is shown in Fig. 4. As clearly seen, the increase of the noise level in the stochastic bistable system (2) leads to the decrease of effective diffusion constant (8) which passes through a minimum. That means the growth of the degree of phase coherence between the output and input signals at an optimal value of noise intensity. As was mentioned above, the classical phenomenon of synchronization is characterized by the presence of the regions of phase and frequency locking (Arnold tongues). To construct such a region in our case we define the condition of effective synchronization as

$$D_{\mathrm{eff}}\leqslant \frac{4\pi^2}{Tn}, \tag{9}$$

where $n \gg 1$ is the number of quasiperiods T of chaotic oscillations. In our study we took $n=100$, e.g., the system (2) can be considered as effectively synchronized if the instantaneous phases are locked in the course of at least 100 quasiperiods of the input signal. The region of phase locking is shown in Fig. 5. It possesses a tonguelike shape that makes the analogy with the classical synchronization more obvious. The threshold character of synchronization is also clearly seen. The calculation of the residence-time distribution [17] has shown that in the regime of synchronization it contains a single peak at the time that corresponds to half of the quasi-

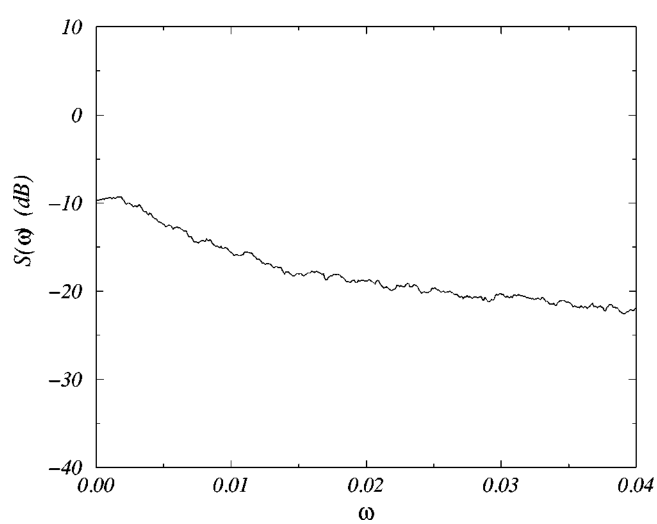

FIG. 6. The power spectrum of oscillations in the Lorenz system.

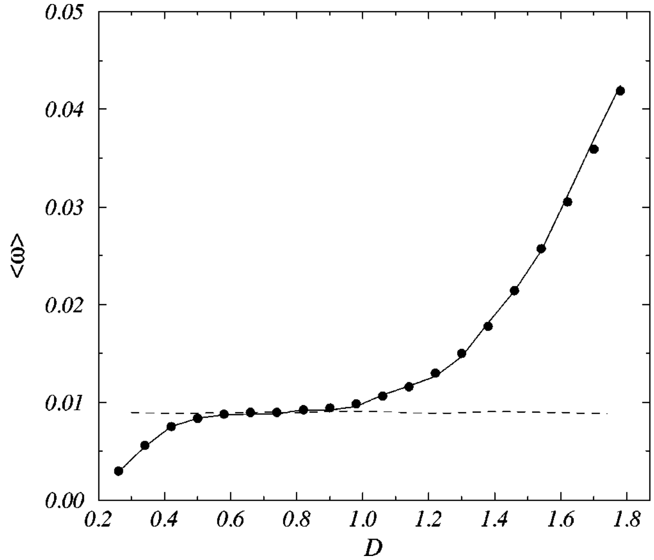

FIG. 7. Mean frequency (6) (solid line) and mean frequency of switchings (11) (symbols) in a stochastic bistable system versus noise intensity. Dashed line is the value of the mean frequency of chaotic oscillations in the Lorenz system. Other parameters are $\alpha=5$, $\beta=1$, $k=0.21$, $\mu(t)=y_1(t)$, $\tau=0.005$.

period. Moreover, this peak remains alone in a finite region of noise intensities, which is in good agreement with the results reported in [29].

III. SYNCHRONIZATION OF SWITCHINGS BY THE BROADBAND CHAOTIC SIGNAL

Now, let us consider the case where the spectrum of the input chaotic signal does not contain any sharp peaks. For this purpose we take the y_1 variable of the Lorenz system: $dy_1/dt=10(y_2-y_1)\tau$, $dy_2/dt=(28y_1-y_2-y_1y_3)\tau$, $dy_3/dt=(y_1y_2-8/3y_3)\tau$, as the input signal $\mu(t)$. The power spectrum of chaotic oscillations $y_1(t)$ is pictured in Fig. 6. As is well known, for these values of parameters the

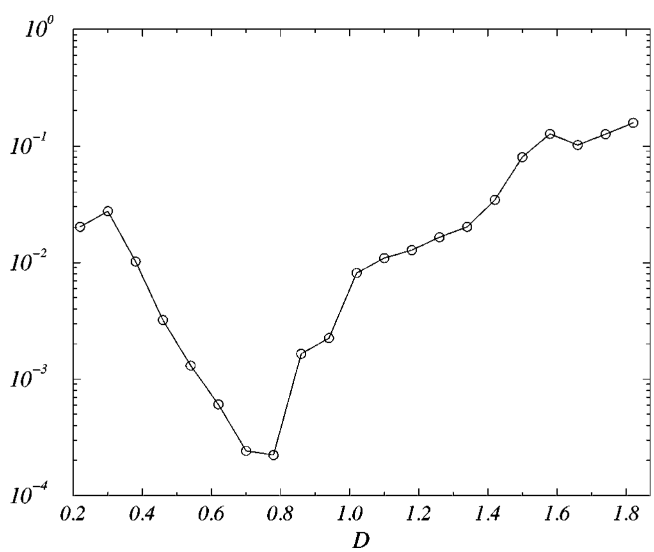

FIG. 8. The dependence of the effective diffusion constant D_{eff} vs noise intensity in the case of an input signal from the Lorenz system. Other parameters are the same as in Fig. 7.

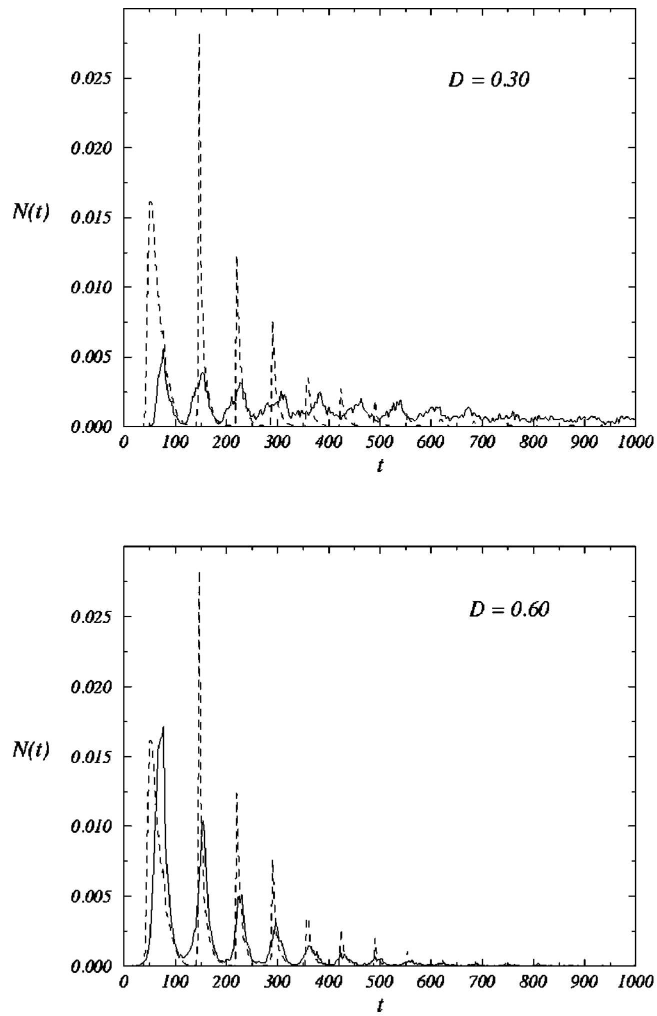

FIG. 9. Residence-time distributions for indicated values of noise intensity. Other parameters are the same as in Fig. 7.

Lorenz-type attractor exists in the phase space and the dynamics of this system can be considered as a random process of switchings between two states [34,35]. For this reason it is natural to introduce the instantaneous phase of the input signal and of the response through the return times in the secant plane $y_1=0$:

$$\Phi(t)=2\pi\frac{t-t_k}{t_{k+1}-t_k}+2\pi k, \quad t_k<t<t_{k+1}. \tag{10}$$

The time dependence of the instantaneous phase is a piecewise-linear function in this case. The instantaneous frequency $\omega(t)=2\pi/T(t)$ is constant during the time interval $t_k<t<t_{k+1}$, while the mean frequency for this definition is equivalent to the mean switching frequency of the system:

$$\langle\omega\rangle=\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}\frac{2\pi}{t_{k+1}-t_k}. \tag{11}$$

The results of our calculations have shown that the instantaneous phase difference demonstrates exactly the same behavior as in the previous cases, e.g., the noise-enhanced phase locking takes place again (not shown). It should be noted that using two different definitions of the instantaneous phases

(5) and (10) gives the same results for averaged quantities. As is seen from Fig. 7, the mean frequency of stochastic oscillations coincides with the mean frequency of chaotic oscillations in some range of noise intensity, which means *the locking of mean frequency of chaotic oscillations*. In other words, the switchings in the Lorenz system synchronize the switchings in the stochastic bistable system at an optimal value of noise intensity. To confirm these results we have calculated the effective diffusion constant. Its dependence versus noise intensity is shown in Fig. 8. As in the previous case of an input signal from the Rössler system, the increase of noise intensity leads to the growth of phase coherence between the switchings in a stochastic bistable system and chaotic oscillations. It manifests in the decrease of D_{eff} in some range of the noise intensity. This fact means that the growth of noise intensity causes the increase of duration of the time intervals in the course of which the switchings in system (2) remain synchronized with the switchings in the Lorenz system.

Since the dynamics of the Lorenz system can be considered as the random process of switchings between two metastable states, then it is natural to use the residence time distributions [17] of input and output signals to analyze the temporal structure of the response for different values of the noise intensity. The results of our calculations of residence-time distributions are reported in Fig. 9. It is important to note that the residence-time distribution of the chaotic input signal has a clearly distinguishable structure (dashed line in Fig. 9) that is caused by the multifractality of the Lorenz system [36]. The increase of the noise intensity makes this structure more ''visible'' at the output of the stochastic resonator (solid line in Fig. 9). At an optimal noise level the structures of the input and output signals nearly coincide. Thus, the above results clearly show that stochastic bistable system can be effectively synchronized by the broadband chaotic signal.

IV. CONCLUSION

We have studied the dynamics of a stochastic bistable system driven by a chaotic input signal. The effect of synchronization of a switching process in a noisy bistable system by the input chaotic signal is found. We have introduced the instantaneous phases of an input chaotic signal and of the response and described this effect in terms of the classical theory of oscillations. The instantaneous phase difference demonstrates behavior that is similar to that of a synchronized self-sustained oscillator. However, in contrast to the classical case, noise plays a positive role enhancing the phase coherence between the input chaotic signal and the response that manifests in the *locking of instantaneous phases and mean frequencies* of stochastic and chaotic oscillations at an optimal noise level. The mentioned effect of phase synchronization of a stochastic switching process is shown to occur in a finite region of noise intensity. The effective diffusion constant takes its minimal value inside of this region, which demonstrates once more the increase of phase coherence between the output and input signals. The above results can be considered as a generalization of the approach proposed in [29] to the case of ASR, which, as well as SR, manifests itself as a noise-enhanced phase locking phenomenon.

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